Definitions:

Domain - the set of all possible ( ) values of a relation.

Range - the set of all possible ( ) values of a relation.

Relation - a set of ordered pair(s)

Function - a relation in which each domain ( ) value is paired with only one unique range ( ) value.

Vertical line test - an equation defines y as a function of x if and only if every vertical line in the coordinate plane intersects the graph of the equation only once.

Example 1: Determine the domain/range of the following graphs and whether they are a function/relation

Types of Functions:

1. Constant function: - eg. $x = k$

2. Linear: $y = mx + b$ or $f(x) = mx + b$
3. Quadratic Standard form

\[ f(x) = a(x-h)^2 + k \]

General (expanded) form

\[ f(x) = ax^2 + bx + c \quad a, b, c \in \mathbb{R} \quad a \neq 0 \]

Vertex

Axis of symmetry:
if \( a > 0 \), graph opens ______
if \( a < 0 \), graph opens ______

4. Cubic: \( f(x) = ax^3 + bx^2 + cx + d \)

5. Absolute value: \( f(x) = a|x-h| + k \)

6. Radical: \( f(x) = a\sqrt{x-h} + k \)

7. Reciprocal: \( f(x) = \frac{a}{x-h} + k \)
Restrictions on the domain of a functions:

1. Cannot have a negative number inside an even root. \( f(x) = \sqrt{3-x} \)

2. Cannot have zero in a denominator \( f(x) = \frac{8}{x-6} \)

One–to–One Function

A one-to-one function is a function in which every single value of the domain is associated with only one value in the range, and vice-versa.

Horizontal line test - for a one-to-one function:

A function, \( f(x) \), is a one-to-one function of \( x \) if and only if every horizontal line in the coordinate plane intersects the function only once at most.

Example 2: Determine whether the following relations are functions, one-to-one functions or neither

Practice: Pg 9 & 10 # 1, 2 and 3
Functions are number generators. When you put a value for the domain in the function, you will get the resulting value in the range.

\[ f(x) = x^2 + 5 \]

And just like numbers, functions can be added, subtracted, multiplied and divided.

Use the following functions, \( f(x) = 2x - 1 \) & \( g(x) = x^2 - 4 \) to determine:

a) Sum: \( (f + g)(x) = f(x) + g(x) \)

b) Difference: \( (f - g)(x) = f(x) - g(x) \)

Product: \( (fg)(x) = f(x) \cdot g(x) \)

Quotient: \( \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \)

**Note:** The domain of the new function must include the restrictions of the new functions as well as the restriction(s) of the original function(s)

**Example 1:** Given functions below, determine each new combined function and its domain.

\[ f(x) = \frac{1}{x^2}, \quad g(x) = \frac{2}{x + 2}, \quad h(x) = 2x^2 - 5x - 3 \quad i(x) = x^2 - 9, \quad j(x) = x^2, \quad k(x) = \sqrt{x} \]

a) \( (gi)(-3) \)

b) \( \left( \frac{f}{g} \right)(-4) \)
Example 1 continued... Given functions below, determine each new combined function and its domain.

\[ f(x) = \frac{1}{x^2}, \quad g(x) = \frac{2}{x+2}, \quad h(x) = 2x^2 - 5x - 3 \quad i(x) = x^2 - 9, \quad j(x) = x^2, \quad k(x) = \sqrt{x} \]

c) \( (g - f)(x) \) 

d) \( \left( \frac{gk}{i} \right)(x) \)

e) \( \left( \frac{1}{k} \right) \left( \frac{1}{k} \right)(x) \) 

f) \( \left( \frac{f}{x} \right)(x) \)
Example 1 continued... Given functions below, determine each new combined function and its domain.

\[ f(x) = \frac{1}{x^2}, \quad g(x) = \frac{2}{x + 2}, \quad h(x) = 2x^2 - 5x - 3 \quad i(x) = x^2 - 9, \quad j(x) = x^2, \quad k(x) = \sqrt{x} \]

\[ g) \quad \left( \frac{g}{f} \right)(x) - \left( \frac{h}{i} \right)(x) \]

\[ h) \quad \left[ g(h - i) \right](x) \]
Example 2: Use the graph of \( f(x) \) and \( g(x) \) to graph the following functions.

a) \((f - g)(x)\)

b) \(\left(2f - \frac{1}{2}g\right)(x)\)

Practice: Pg 13-16 # 1, 2, 3 (a, b, c, d, i, j), 4 (a, d), 5b, 6c, 7d, 8e
A function can be considered like a machine. It has an input value \( x \) and generates an output, \( y \), or \( f(x) \).

\[ f(x) = 3x - 4 \]

A function can also be “input” into another function. This would then generate a composite function.

Using the functions \( f(x) = 3x - 4 \) and \( g(x) = x^2 + 2 \), determine:

\[ f(g(x)) \]
\[ g(f(x)) \]

**Notation:** \( f(g(x)) = (f \circ g)(x) = f \circ g \)

With composite functions, the output of one function becomes the input of another function(s).

When determining \( (f \circ g)(x) \) the output of function \( g(x) \) becomes the input of the function \( f(x) \).

**Example 1:** Given functions below, determine each composite function.

\[ f(x) = \frac{1}{x^2}, \quad g(x) = \frac{2}{x + 2}, \quad h(x) = 2x^2 - 5x - 3, \quad i(x) = x^2 - 9, \quad k(x) = \sqrt{x} \]

a) \( (h \circ g)(-3) \)

b) \( (k \circ g \circ f)(-2) \)
Example 1 continued.......  

\[ f(x) = \frac{1}{x^2}, \quad g(x) = \frac{2}{x+2}, \quad h(x) = 2x^2 - 5x - 3, \quad i(x) = x^2 - 9, \quad k(x) = \sqrt{x} \]

c) \((h \circ i)(x)\)  
d) \(i(h(x))\)

---

**Domain of a Composite Function**

The domain of \((f \circ g)(x)\) has both the original restriction(s) of \(g(x)\) as well as the domain restrictions of the final composite function \((f \circ g)(x)\).

---

**Example 2:** Given the functions \(f(x) = \frac{4}{x}\), \(g(x) = \frac{2}{x-3}\) determine each composite function and its domain.

a) \((f \circ g)(x)\)  
b) \((g \circ f)(x)\)  
c) \((g \circ g)(x)\)
Example 3: Calculus Application: Compute \( \frac{f(x + h) - f(x)}{h} \), for \( f(x) = 3x^2 - 2x \)

Decomposing a Function:

To decompose a function, look for a simple function that could be the input of another simple function.

Example 4: Find two functions \( f(x) \) and \( g(x) \) such that \( h(x) = (f \circ g)(x) \)

a) \( h(x) = \sqrt{2x^2 + 1} - 5 \) 

b) \( h(x) = \frac{2}{3x + 4} \)
Pre-Calculus Mathematics 12 - 1.3 – Composite Functions

The output of a composite function can also be determined without having the actual functions as long as you have the corresponding graphs of the functions.

Example 5: Given the functions $f(x)$ and $g(x)$, determine:

a) $(f \circ g)(-4)$

b) $(g \circ f)(3)$

c) $f((g)(-8))$

d) $(g \circ g)(5)$

Practice: Pg 22 – 26 #1 (a, b, c), 2(a, b, g), 3(a, f), 4(b, h), 5a, 6d, 7(a, m), 10b, 11
**Vertical translations** (shifting the graph up or down)  
form:  \( y = f(x - h) + k \)

Square Root  
\[ y = \sqrt{x} - 2 \]

Absolute Value  
\[ y = |x| + 3 \]

So for  \( y = f(x - h) + k \)

If \( k \) is positive \( \rightarrow \) _________________  
If \( k \) is negative \( \rightarrow \) _________________

**Example 1**: Given the graph of \( y = f(x) \) below, describe the transformation applied graph \( y = f(x) - 2 \), and map the coordinates of the image points.
**Horizontal Translations** (shifting the graph left or right) form: \( y = f(x - h) \)

Graph: \( y = x^2 \); \( y = (x - 3)^2 \); \( y = (x + 2)^2 \)

For \( y = f(x - h) \)

If \( h \) is positive \( \rightarrow \) \__________________  If \( h \) is negative \( \leftarrow \) \__________________

\__________________

**Example 2:** Given the graph of \( y = f(x) \) below, describe the transformation applied graph \( y = f(x - 2) \), and map the coordinates of the image points.
**Example 3:** Given the graph of \( y = f(x) \) below, describe the transformation applied to the graph \( y = f(x - 3) - 4 \), and generalize the coordinate transformation.

![Graph of \( y = f(x) \)](graph.png)

**Example 4:** Given the function \( y = f(x) \) below, describe the transformation applied to each of the functions below.

\[
a) \quad y = f(x + 3) - 2 \\
\]

\[
b) \quad y = f(x - 5) + 6 \\
\]

\[
c) \quad y + 1 = f(x + 5) \\
\]

\[
d) \quad y - 4 = f(x - 7) - 7 \\
\]

---

**Practice:** Complete the questions on the next page, the solutions are provided
Practise

1. For each function, state the values of \( h \) and \( k \), the parameters that represent the horizontal and vertical translations applied to \( y = f(x) \).
   a) \( y = 5 = f(x) \)
   b) \( y = f(x) - 4 \)
   c) \( y = f(x + 1) \)
   d) \( y + 3 = f(x - 7) \)
   e) \( y = f(x + 2) + 4 \)

2. Given the graph of \( y = f(x) \) and each of the following transformations,
   • state the coordinates of the image points \( A', B', C', D' \) and \( E' \).
   • sketch the graph of the transformed function
   a) \( g(x) = f(x) + 3 \)
   b) \( h(x) = f(x - 2) \)
   c) \( s(x) = f(x + 4) \)
   d) \( t(x) = f(x) - 2 \)

3. Describe, using mapping notation, how the graphs of the following functions can be obtained from the graph of \( y = f(x) \).
   a) \( y = f(x + 10) \)
   b) \( y + 6 = f(x) \)
   c) \( y = f(x - 7) + 4 \)
   d) \( y - 3 = f(x - 1) \)

4. Given the graph of \( y = f(x) \), sketch the graph of the transformed function. Describe the transformation that can be applied to the graph of \( f(x) \) to obtain the graph of the transformed function. Then, write the transformation using mapping notation.
   a) \( r(x) = f(x + 4) - 3 \)
   b) \( s(x) = f(x - 2) - 4 \)
   c) \( t(x) = f(x - 2) + 5 \)
   d) \( v(x) = f(x + 3) + 2 \)

5. For each transformation, identify the values of \( h \) and \( k \). Then, write the equation of the transformed function in the form \( y - k = f(x - h) \).
   a) \( f(x) = \frac{1}{x} \), translated 5 units to the left and 4 units up
   b) \( f(x) = x^4 \), translated 8 units to the right and 6 units up
   c) \( f(x) = |x| \), translated 10 units to the right and 8 units down
   d) \( y = f(x) \), translated 7 units to the left and 12 units down

1. a) \( h = 0, k = 5 \)  b) \( h = 0, k = -4 \)  c) \( h = -1, k = 0 \)
   d) \( h = 7, k = -3 \)  e) \( h = -2, k = 4 \)
2. a) \( A'(-4, 1), B'(-3, 4) \)
   b) \( A'(-2, -2), B'(-1, 1), C'(-1, 1), D'(1, 2), E'(2, 2) \)
   c) \( A'(-8, -2), B'(-7, 1), C'(-5, 1), D'(-3, -1), E'(-2, -1) \)
   d) \( A'(-4, -1), B'(-3, -1), C'(-1, -1), D'(1, -3), E'(2, -3) \)
3. a) \( (x, y) \rightarrow (x - 10, y) \)
   b) \( (x, y) \rightarrow (x, y - 6) \)
   c) \( (x, y) \rightarrow (x + 7, y + 4) \)
   d) \( (x, y) \rightarrow (x + 1, y + 3) \)
4. a) \( r(x) = f(x + 4) - 3 \)
   b) \( s(x) = f(x - 2) - 4 \)
   c) \( t(x) = f(x - 2) + 5 \)
   d) \( v(x) = f(x + 3) + 2 \)
   a) \( a \) vertical translation of 5 units up and a horizontal translation of 2 units right;
   \( (x, y) \rightarrow (x + 2, y + 5) \)
   a) \( a \) vertical translation of 5 units up and a horizontal translation of 2 units right;
   \( (x, y) \rightarrow (x + 2, y + 5) \)
Pre-Calculus Mathematics 12 - 1.4 – Transformations of Graphs Part 2

**Goals:**
1. Identify and apply a reflection to a function with or without a graph
2. Identify and apply a vertical compression/expansion to a function
3. Identify and apply a horizontal compression/expansion to a function

**Reflections in the Coordinate Axis** (flipping the graph over the x or y axis)

\[ y = -\sqrt{x} \quad \quad \quad y = \sqrt{-x} \]

So for \( y = f(x) \)

If \( y = -f(x) \) \( \rightarrow \) ____________

If \( y = f(-x) \) \( \rightarrow \) ____________

---

**Example 1:** Given the graph of \( y = f(x) \) below, describe the transformation applied to the graph \( y = f(-x) \), and map the general coordinates of the image points.
Vertical Expansions/Compressions (stretching the graph in y direction)

Graph: \( y = \sqrt{x} \) \( y = 2\sqrt{x} \) \( y = \frac{1}{2}\sqrt{x} \)

For \( y = af(x) \)

If \(|a| > 1 \rightarrow \) 

If \(|a| < 1 \rightarrow \)

Example 2: Given the graph of \( y = f(x) \) below, describe the transformation applied to the graph \( y = \frac{1}{3}f(x) \), and map the general coordinates of the image points.
Horizontal Compressions/Expansions (stretching the graph in x direction)

Graph: \( y = x^2 \) \[ y = (2x)^2 \] \[ y = \left(\frac{1}{2}x\right)^2 \]

For \( y = f(bx) \)

If \(|b| > 1\) \(\rightarrow\) ________________  
If \(|b| < 1\) \(\rightarrow\) ________________

Example 3: Given the graph of \( y = f(x) \) below, describe the transformation applied to the graph \( y = f(2x) \), and map the general coordinates of the image points.
Example 4: Given the function $y = f(x)$ below, describe the transformation applied to each of the functions below.

a) $y = -2f(x-4) + 6$

b) $y = f(-3x - 12) - 4$

c) $2y + 10 = f(5 - x)$

Example 5: Given the graph of $y = f(x)$, sketch the graph of $y = \frac{1}{2} f\left(\frac{1}{3}x\right)$.

Example 6: If (-3, 4) is a point on the graph $y = f(x)$ what must be the point on the graph $y + 7 = -5f(2x-4) + 3$.

Practice: Pg 34 # 1 (a, c, e), 2 (a, c, e, f), 3, 4, 6, 9c, 10 (b, d, f), 11d, 12(c, g)
Two functions are inverse functions if one function “undoes” what the other function “does”.

Inverse functions are just a reflection across the line $y = x$. In order for both a function $f(x)$ and its inverse $f^{-1}(x)$ to qualify as functions, $f(x)$ must be a one-to-one function.

**Determining the inverse of a function:**

1. Verify that $f(x)$ is a one-to-one function. (If not, its inverse is not a function.)
2. Replace $f(x)$ with $y$ and exchange all $x$’s and $y$’s.
3. Solve for $y$.
4. Replace the new $y$ with $f^{-1}(x)$. (Only if $f^{-1}(x)$ is actually a function.)

**Example 1:** If $f(x) = 3x - 2$, determine the inverse of the function.

**Example 2:** If $g(x) = \frac{x}{3x - 1}$, determine the inverse of the function.
Example 3: If \( h(x) = x^2 - 3 \), determine the inverse of the function.

Determining if two functions are inverses: (‘undo’ each other)

Two functions \( f(x) \) and \( g(x) \) are inverses of each other if and only if

\[
(f \circ g)(x) = x \quad \text{for every value of } x \text{ in the domain of } g \quad \text{and} \\
(g \circ f)(x) = x \quad \text{for every value of } x \text{ in the domain of } f.
\]

Example 4: Determine whether \( f(x) = \frac{x}{2x-3} \) and \( g(x) = \frac{3x}{2x-1} \) are inverses of each other.
Graphing the inverse of a function

The graph of a function and its inverse are symmetric about the line $y = x$.

$f(x)$ is the reflection of $f^{-1}(x)$ on the line $y = x$ and vice versa.

Example 5: Given the function $f(x) = (x + 2)^2 + 4$, graph the inverse of the function. Determine whether the inverse of the function is a function.

Example 6: If (-3, 4) is a point on the graph $y = f(x)$ what must be the point on the graph $y = 1 - f^{-1}(-x)$.

Practice: Page 44 # 2(a,b,c), 4, 5(a,b,f), 6 (a, c, f), 7(a, b, c)
All of the transformations we have performed can be summarized as follows:

\[ y = f(x) \text{ transforms to } y = af[b(x-h)] + k \]

**Reflections:**
- \( a < 0 \), reflection in the x-axis
- \( b < 0 \), reflection in the y-axis
- \( f^{-1} \) reflection across the line \( y = x \)

**Expansion:**
- \( |a| > 1 \), vertical expansion by a factor of \( |a| \)
- \( |b| < 1 \), horizontal expansion by a factor of \( \frac{1}{|b|} \)

**Compression:**
- \( |a| < 1 \), vertical compression by a factor of \( |a| \)
- \( |b| > 1 \), horizontal compression by a factor of \( \frac{1}{|b|} \)

**Translation:**
- \( k > 0 \), vertical translation \( k \) units up
- \( k < 0 \), vertical translation \( k \) units down
- \( h > 0 \), horizontal translation \( h \) units right
- \( h < 0 \), horizontal translation \( h \) units left

When combining transformations, the reflections/expansions/compressions must occur before the translations.

**Example 1:** Given point \( P(-4, 2) \) on \( y = f(x) \) find the new location for \( P \) on:

\[ y = -f\left(\frac{x}{3}\right) + 2 \]

**Example 2:** Given point \( P(a, b) \) on \( y = f(x) \) find the new location for \( P \) on:

\[ y = -3f(6 - 2x) - 5 \]
Example 3: Graph the following functions:

a) \( y = -\frac{1}{2}|x - 3| + 4 \)

b) \( h(x) = -3\sqrt{-2(x+1)} + 4 \)

c) If \( f(x) = x^2 + 1 \) \( y = -\frac{1}{2} f(2x + 4) - 3 \)
Example 4: Given the graph of \( y = f(x) \), sketch the graph \( y = -2f(x+3) + 1 \)

Practice: Page 51 #2 (a, b, c, e, g, j), 3(a, b, c, d), 6 (a, b, c), 7(a, c, e)